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On the classification of complex tori arising from real Abelian surfaces

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Abstract Let A' be an Abelian surface over $\mathbb{R}$ and denote by A its complexification. We define an intrinsic volume $vol(A)$ of A and show that there are seven possibilities with respect to the rank of $End(A)$ and if $vol(A)$ is rational or not. We prove that each possibility determines the Picard number and the endomorphism algebra of A' and A respectively.

Keywords Real algebraic geometry · Abelian surfaces

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1 Introduction

The article is devoted to the classification of complex tori which arise from Abelian surfaces A' over $\mathbb{R}$. The result is mainly motivated by problems from Diophantine geometry and the theory of transcendental numbers. In these areas, many questions come down to the interaction of the arithmetic of an Abelian variety A' over a number field K and the geometry of the associated torus $T = A(\mathbb{C})$, where A is the complexification of A'. Moreover, next to its arithmetic flavor the particular but important case of real number fields $K \subset \mathbb{R}$, including the field of rational numbers, also gives rise to special problems and conjectures concerning the topology of real points (cf. [4], e.g.).

With this perspective, we obtained a classification scheme for complex Abelian surfaces A in terms of the splitting behavior of A' resp. A, properties of the lattice $\Lambda \subset \mathbb{C}^2$ defining T, and the Picard numbers $\rho(A')$ and $\rho(A) = \rho(T)$. In particular, the following results are proved in an elementary manner:

(1) If A' is simple then $\rho(A')$ equals the rank of $End(A')$. Conversely, if the rank is equal to the Picard number then A' is simple, or A' admits an isogeny of degree ≤ 8 to a product of non-isogenous elliptic curves, each without complex multiplication.

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(2) Assume that A is the complex torus associated to a real surface A' over $\mathbb{R}$ and that A is simple. Choose a polarization H of A which is induced by a line bundle of A' and let $\phi \mapsto \phi'$ be the corresponding Rosati involution on $End(A)$ with subalgebra of fixed elements $End^s(A)$. Then $End(A') \subset End^s(A)$, and $End(A')$ is an order in a real number field of degree $\rho(A') \leq 2$.
(3) In the preceding situation, A has quaternion multiplication precisely if $\rho(A') + 1 = \rho(A)$. In this case, $End(A')$ is an order in a real quadratic number field, and there exists a further order $\mathcal{O} \subset End(A)$ of a real quadratic number field such that $\mathcal{O}$ is not contained in $End(A')$.
(4) Suppose that A' is simple over $\mathbb{R}$, whereas A is a product of elliptic curves. Then the elliptic curves in question are pairwise conjugate, but without CM. They are isogenous if and only if $End(A')$ is an order in a real quadratic number field.

Results as above yield useful criteria for the determination of the field of definition for a given Abelian surface and its ring of endomorphisms and are necessary to understand moduli of polarized Abelian varieties with prescribed endomorphism structure. (2) is known for Abelian varieties over $\mathbb{Q}$ (cf. [2, Proposition 1.3.]). (3) should be compared with a theorem of Shimura (cf. [5]).

2 The main result

2.1 Statement of the theorem

Let A be a complex Abelian variety of dimension $\dim A = g$ which is defined over the reals; that is, $A = A' \times_{\mathrm{spec}\,\mathbb{R}} \mathrm{spec}\,\mathbb{C}$ for a $A' \in \mathbf{Var}_{\mathbb{R}}$. Here, we use the notion of variety in the sense of algebraic geometry. One has a natural inclusion $End(A') \subset End(A)$, and we will say that $End(A)$ *is defined over* $\mathbb{R}$ if equality holds. Next, we define F to be $End(A) \otimes \mathbb{Q}$. We fix a polarization $H \in Pic(A)$ and let $End^s(A) \subset End(A)$ be the subalgebra fixed by the induced Rosati involution. Moreover, we let $\mathcal{T}_A$ be the Lie algebra of the associated complex torus $T = A(\mathbb{C})$.

The exponential mapping $\exp_A : \mathcal{T}_A(\mathbb{C}) \longrightarrow T$ provides an analytic isomorphism $T = \mathcal{T}_A(\mathbb{C})/\Lambda$ where Λ is the lattice equal to the kernel of $\exp_A$. As explained in the next section, $\mathcal{T}_A$ inherits a real structure from A and there is a maximal sublattice $\Lambda_* \subset \Lambda$ which decomposes as $\Lambda_1 \oplus \Lambda_2$ with $\Lambda_1 \subset \mathcal{T}_A(\mathbb{R})$ and $\Lambda_2 \subset i \cdot \mathcal{T}_A(\mathbb{R})$: take $\Lambda_1 = \Lambda \cap \mathcal{T}_A(\mathbb{R})$ and $\Lambda_2 = \Lambda \cap i \cdot \mathcal{T}_A(\mathbb{R})$. Let $\omega_1, \dots, \omega_{2g}$ be a basis of Λ_* over $\mathbb{Z}$ such that

$$\omega_1, \dots, \omega_g \in \Lambda_1, \qquad \omega_{g+1}, \dots, \omega_{2g} \in \Lambda_2.$$

Then, representing $(\omega_1, \dots, \omega_{2g})$ by a real-valued matrix $\mathbf{M}$ with respect to the basis $\omega_1, \dots, \omega_g, i\omega_1, \dots, i\omega_g$, the number

$$vol(A) = |\det \mathbf{M}|$$

depends only on A, because another basis of Λ_* with the above properties results from the original one by multiplication with a matrix in $\mathbf{Sl}_{g\times g}(\mathbb{Z}) \oplus \mathbf{Sl}_{g\times g}(\mathbb{Z})$. Note that $vol(A)$ is in fact the volume of a fundamental domain for Λ with respect to a suitably chosen norm on $\mathcal{T}_A(\mathbb{C})$. Our theorem below asserts that it plays an important role in the geometry of A whether $vol(A)$ is rational or not. In particular, by Lemma 3.8 we have $\rho(A) = \rho(A') + \chi_{\mathbb{Q}}(vol(A))$ if $g = 2$. Here, $\chi_{\mathbb{Q}} : \mathbb{R} \longrightarrow \{0, 1\}$ is the function which vanishes precisely for irrational numbers.

From now on, let A be a complex Abelian *surface* defined over the reals. Denote by $A(\mathbb{R})$ the Lie group of points with real coordinates and by $A(\mathbb{R})^o$ its identity component. Write $\mathcal{B}$ for the set of elliptic curves $B \subset A$ which are defined over $\mathbb{R}$ and with the property that the neutral elements e_B and e_A are equal. An elliptic curve $B \subset A$ lies in $\mathcal{B}$ if $B(\mathbb{R})^o$, the unit component of $B(\mathbb{C}) \cap A(\mathbb{R})^o$ with respect to the analytic topology, has real dimension 1 and $e_B = e_A$. In fact, $B(\mathbb{C})$ is the Zariski-closure of $B(\mathbb{R})$, so that B is then defined over the reals. Next, let $\mathcal{G}$ be the set of compact real Lie subgroups $G \subset A(\mathbb{R})^o$ with dimension 1. One has a natural mapping $\Phi : \mathcal{B} \longrightarrow \mathcal{G}$, where $\Phi(B)$ is defined to be $B(\mathbb{R})^o$. Finally, $\sigma \in Gal(\mathbb{C}|\mathbb{R})$ is the complex conjugation, and for a $\mathbb{C}$-variety $V \to \operatorname{spec}\mathbb{C}$ we define $V^\sigma \in \mathbf{Var}_\mathbb{C}$ to be V together with the structure morphism $V \to \operatorname{spec}\mathbb{C} \xrightarrow{\sigma} \operatorname{spec}\mathbb{C}$.

With this our *main result* takes the form of a classification table, where we divided the appearing cases into seven alternatives. In the statement, CM and RM mean real and complex multiplication, that is, that F contains an imaginary resp. totally real quadratic number field. We also recall that a quaternion algebra $\mathbb{Q} + \mathbb{Q}i + \mathbb{Q}j + \mathbb{Q}ij$ with $i^2, j^2 \in \mathbb{Q}_*$, $ij = -ji$, is said to be *totally indefinite* if i^2 and j^2 have different signs.

Theorem 2.1 *With preceding notations, the following table together with the below descriptions cover all possibilities.*

Alternative	1a	1b	1c	2a	2b	2c	3a	3b
$\mathcal{B}$	*infinite*	*infinite*	*finite*	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\mathbf{\Phi}$	*bijective*	*bijective*	*not bij.*	–	–	–	–	–
$\rho(\mathbf{A})$	4	3	2	1	2	3	2	3
$\chi_\mathbb{Q}(vol(\mathbf{A}))$	1	0	0	0	0	1	1	1
$[F:\mathbb{Q}]$	8	4	2–4	1	2	4	2	4
CM	*yes*	*yes*	*iff* $[F:\mathbb{Q}] \geq 3$	*no*	*no*	*yes*	*no*	*yes*
RM	*yes*	*yes*	*no*	*no*	*yes*	*yes*	*no*	*yes*
$\mathbf{A}$ *simple over* $\mathbb{C}$	*no*	*no*	*no*	*yes*	*yes*	*yes*	*no*	*no*
$\mathbf{A}'$ *simple over* $\mathbb{R}$	*no*	*no*	*no*	*yes*	*yes*	*yes*	*yes*	*yes*

Alternative 1a. *A' is isogenous over $\mathbb{R}$ to a power $B' \times B'$ of an elliptic curve B' such that B, the complexification of B', has CM. If this happens then the components of ω_3, ω_4 with respect to an arbitrary $\mathbb{R}$-basis $\omega_1, \omega_2 \subset \Lambda$ of $\mathcal{T}_A(\mathbb{R})$ generate an imaginary quadratic number field. Also, A is isomorphic to a product of elliptic curves which are not necessarily defined over $\mathbb{R}$.*

Alternative 1b. *A' is isogenous over $\mathbb{R}$ to a power $B' \times B'$ of an elliptic curve B' such that B does not admit CM. In this situation, $End(A)$ is defined over $\mathbb{R}$.*

Alternative 1c. *A admits an isogeny defined over $\mathbb{R}$ to a product of pairwise non-isogenous elliptic curves such that the isogeny has kernel isomorphic to $\mathbb{Z}/2^k\mathbb{Z}$ where $4-k$ equals the number of topological components of the real Lie group $A(\mathbb{R})$.*

Alternative 2b. *F is defined over $\mathbb{R}$ and is a totally real quadratic number field.*

Alternative 2c. *We have $End(A') = End^s(A)$. $End(A')$ is an order in real quadratic number field, and F is a totally indefinite quaternion algebra which is not defined over $\mathbb{R}$. This happens precisely if T is isogenous to a torus with period matrix*

$$\begin{pmatrix} 1 & 0 & i\alpha & i\beta \\ 0 & 1 & i\gamma & -i\alpha \end{pmatrix}$$

such that $\alpha, \beta, \gamma \in \mathbb{R}$ satisfy $\alpha^2 + \beta\gamma \in \mathbb{Q}^*$.

Alternative 3a. *End*(A') *is isomorphic to* $\mathbb{Z}$, *and* A *is isogenous to a product* $B \times B^\sigma$ *for an elliptic curve* B *which is not defined over* $\mathbb{R}$ *and without CM. Here,* B *and* B^σ *are not isogenous.*
Alternative 3b. *End*(A') *is an order in a real quadratic number field and is properly contained in* $End^s(A)$. A *is isogenous to a product* $B \times B^\sigma$ *for an elliptic curve* B. *Thereby,* B *and* B^σ *are isogenous, but not defined over* $\mathbb{R}$ *and without* CM. *This happens precisely if* T *is isogenous to a torus with period matrix*

$$\left(\mathbf{1}_2, \quad \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^\sigma \end{pmatrix}\right)$$

such that $|\alpha|^2 \in \mathbb{Q}$, *but* $\operatorname{Re}\alpha \notin \mathbb{Q}$.

2.2 Existence in the described cases

It is clear that all possibilities of Alternative 1 indeed appear. As to Alternative 2a, infinitely many examples are given by $A(p, q)$, where for two different prime numbers p and q the symbol $A(p, q)$ denotes the Abelian surface subject to the period matrix

$$\Pi_{p,q} = \left(\mathbf{1}_2, \quad i \cdot \begin{pmatrix} \sqrt{p} & 1 \\ 1 & \sqrt{q} \end{pmatrix}\right).$$

Examples of absolutely simple Abelian surfaces over $\mathbb{Q}$ with F a real quadratic number field or totally indefinite quaternion algebra are provided by two-dimensional simple factors of modular Jacobian varieties $J_1(N)$. We refer to [2], e.g. It is part of the conjecture of Serre-Ribet that each Abelian surface over $\mathbb{Q}$ with real or potential quaternion multiplication is a simple factor of $J_1(N)$. An example for Alternative 3a is given in [3, Example 25]. Finally, the existence of surfaces as in Alternative 3b follows from the statement of the theorem.

2.3 Organization of the paper

The category of complex Abelian varieties is equivalent to the category of pairs (V, Λ) where V is a complex vector space and $\Lambda \subset V$ is a lattice subject to Riemann's relations. In fact, the largest part of the proof of the theorem relies on elementary linear algebra, despite its nontrivial consequences. Thereby, many arguments use the restriction to 2 complex (or 4 real) variables, and are thus not generalizable to higher dimensions in a direct manner.

For convenience of the reader, we prepared a rather detailed exposition on generalities of complex Abelian varieties defined over $\mathbb{R}$ (Sect. 3). The proof of our main result is then given in Sect. 4. We obtain our table in Alternative 1 by supposing A' not simple over $\mathbb{R}$ (Sect. 4.1). The proof of the remaining part is subdivided into two cases: A is simple over $\mathbb{C}$ (Alternative 2, Sect. 4.3) or not (Alternative 3, Sect. 4.4).

3 Abelian varieties over $\mathbb{R}$

In this section, we summarize some basic facts concerning complex Abelian varieties defined over $\mathbb{R}$. The most comprehensive source for general complex Abelian varieties is the book of Birkenhake and Lange [1]. All stated results for Abelian varieties over $\mathbb{R}$ can be found in the first two chapters of J. Huisman's thesis [3].

3.1 Generalities

(1) A complex projective variety $V \in \mathbf{Var}_{\mathbb{C}}$ is said to be defined over a subfield $K \subset \mathbb{C}$ if $V = V' \times_{\mathrm{spec} K} \mathrm{spec}\, \mathbb{C}$ for a variety $V' \in \mathbf{Var}_K$. This comes down to the following. V admits an embedding into the projective space $\mathbf{P}^N(\mathbb{C})$ with the property that the ideal of the embedded variety is generated by polynomials in $K[X_0, \ldots, X_N]$. For a specified projective embedding, we write $V(K)$ for the set of closed points with coordinates in K. Note that in general $V(K)$ does not coincide with the closed points of $V'/\,\mathrm{spec}\, K$. This is the case if K is algebraically closed. In particular, $V(\mathbb{C})$ is the underlying set of the corresponding complex manifold.

(2) An Abelian variety $A \in \mathbf{Var}_{\mathbb{C}}$ is said to be defined over K if the morphisms constituting the underlying group structure can be realized by polynomials with coefficients in K and the neutral element e_A is a closed point in $A(K)$.

(3) If A is a complex Abelian variety defined over K then $T = A(\mathbb{C})$ is a complex torus. T admits an exponential mapping $\exp_A : \mathcal{T}_A(\mathbb{C}) \longrightarrow T$ from the tangent space $\mathcal{T}_A(\mathbb{C})$ at the neutral element. Thereby, $\mathcal{T}_A(\mathbb{C})$ is in fact $V(\mathbb{C})$ for a variety $V \in \mathbf{Var}_{\mathbb{C}}$ defined over K, namely for the space of derivations of the local ring $\mathcal{O}_{e,A}$ at the neutral element. In particular, in an analytic neighbourhood U of $e_A \in T = A(\mathbb{C})$ the analytic coordinate functions of $\exp_A$ are given by complex power series with coefficients in K. We will write $\mathcal{T}_A(K)$ for $V(K)$ if the set of K-valued points is considered.

(4) Let $\sigma \in Gal(\mathbb{C}|\mathbb{R})$ be the complex conjugation. σ then provides a real-analytic mapping $^{\sigma} : \mathbf{P}^N(\mathbb{C}) \longrightarrow \mathbf{P}^N(\mathbb{C})$, which fixes all varieties that are embedded over $\mathbb{R}$ in $\mathbf{P}^N(\mathbb{C})$.

(5) It results that $A(\mathbb{R})$, where A is defined over $\mathbb{R}$, is closed in T with respect to the analytic topology. Hence, it is a closed subgroup of the real Lie group T. Further, letting $A(\mathbb{R})^o$ denote its identity component, it holds that the restriction of $\exp_A$ to $\mathcal{T}_A(\mathbb{R})$ is naturally identified with the exponential mapping of $A(\mathbb{R})^o$. Corresponding to this, we have a natural identification

$$\mathcal{T}_A(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} = \mathcal{T}_A(\mathbb{C}). \tag{3.1}$$

(6) The last observation can be strenghtened in the following way. Let U be as in (3) and write $\mathcal{T}_A(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} = \mathcal{T}_A(\mathbb{C})$. Then $Gal(\mathbb{C}|\mathbb{R})$ acts on $\mathcal{T}_A(\mathbb{C})$ by $(v \otimes c)^{\sigma} = v \otimes c^{\sigma}$ for $v \in \mathcal{T}_A(\mathbb{R})$ and $c \in \mathbb{C}$. After eventually shrinking U, one can arrange that $U^{\sigma} = U$. Now, from the fact that $\exp_A$ is represented by real power series, one readily derives that $\exp_A(v^{\sigma}) = \exp_A(\omega)^{\sigma}$ for $v \in U$. And since $\mathcal{T}_A(\mathbb{C}) = \bigcup_{n>0} n \cdot U$, we finally get that in fact

$$\forall v \in \mathcal{T}_A(\mathbb{C}) : \exp_A(v^{\sigma}) = \exp_A(v)^{\sigma}. \tag{3.2}$$

(7) Let $G \subset T$ be a closed connected subgroup which is nontrivial. G is then a real Lie subgroup of T with positive real dimension. The tangent space of G at the neutral element is identified with a real subspace $\mathcal{T}_G \subset \mathcal{T}_A(\mathbb{C})$ such that the restriction of $\exp_A$ to $\mathcal{T}_G$ is the exponential mapping of G. One can ask if $\mathcal{T}_G$ is closed under multiplication of $i \in \mathbb{C}$. This is necessarily true if G is $B(\mathbb{C})$ for an Abelian subvariety $B \subset A$. Conversely, a subspace $\mathcal{T} \subset \mathcal{T}_A(\mathbb{C})$ which is closed under the action of $i \in \mathbb{C}$ is the tangent space of a complex subtorus $B(\mathbb{C})$ precisely if the image $\exp_A(\mathcal{T})$ is compact in T.

(8) Again, let A be a complex Abelian variety defined over $\mathbb{R}$. Write g for $\dim A$. Then, according to the above, $\Lambda := \ker \exp_A$ is a lattice of $\mathcal{T}_A(\mathbb{C})$ which is preserved by complex conjugation: $\Lambda^\sigma = \Lambda$. It follows from (3.1) that

$$\Lambda_* := \big(\Lambda \cap \mathcal{T}_A(\mathbb{R})\big) \oplus \big(\Lambda \cap i \cdot \mathcal{T}_A(\mathbb{R})\big)$$

is a subgroup of finite index in Λ. In fact, both groups are not identical in general, but the following can be achieved. By [3, Proposition 70], there exists a basis $\omega_1, \ldots, \omega_g$ of $\Lambda \cap \mathcal{T}_A(\mathbb{R})$ and $\omega_{g+1}, \ldots, \omega_{2g}$ of $\Lambda \cap i \cdot \mathcal{T}_A(\mathbb{R})$ together with a number $k \in \{0, 1, 2\}$, which is completely determined by Λ, such that

$$\Lambda = \sum_{0 \le i \le k} \frac{1}{2}\mathbb{Z}(\omega_i + \omega_{2+i}) + \sum_{k < i \le g} (\mathbb{Z}\omega_i + \mathbb{Z}\omega_{i+2}). \tag{3.3}$$

k is termed the *degree of connectedness of* Λ and it holds that $A(\mathbb{R})/A(\mathbb{R})^o = \mathbb{Z}/2^{2-k}\mathbb{Z}$ (isomorphism of groups). In particular, we deduce

$$\Lambda_* \subset \Lambda \subset \frac{1}{2}\Lambda_*. \tag{3.4}$$

(9) Conversely, if $\Lambda \subset \mathbb{C}^g$ is a lattice satisfying $\Lambda^\sigma = \Lambda$ then the complex torus $\mathbb{C}^g/\Lambda$ inherits the natural action of $Gal(\mathbb{C}|\mathbb{R})$. If $\mathbb{C}^g/\Lambda$ is in addition a projective complex torus with corresponding Abelian variety $A \in \mathbf{Var}_\mathbb{C}$ then A is defined over $\mathbb{R}$. An analogous assertion holds true for homomorphisms between Abelian varieties. Let $A_1', A_2' \in \mathbf{Var}_\mathbb{R}$ be Abelian varieties with associated tori $T_1 = \mathcal{T}_{A_1}(\mathbb{C})/\Lambda_1$ and $T_2 = \mathcal{T}_{A_2}(\mathbb{C})/\Lambda_2$, and choose $\mathbb{C}$-bases $\omega_1, \ldots, \omega_{g_1} \in \Lambda_1 \cap \mathcal{T}_{A_1}(\mathbb{R})$ of $\mathcal{T}_{A_1}(\mathbb{C})$ and $\omega_1', \ldots, \omega_{g_2}' \in \Lambda_2 \cap \mathcal{T}_{A_2}(\mathbb{R})$ of $\mathcal{T}_{A_2}(\mathbb{C})$. Consider the analytic representation

$$\rho_a : Hom(A_1, A_2) \longrightarrow \mathbf{Mat}_{g_2 \times g_1}(\mathbb{C})$$

with respect to these bases. Then

$$\rho_a\big(Hom(A_1', A_2') \otimes \mathbb{Q}\big) = \rho_a\big(Hom(A_1, A_2) \otimes \mathbb{Q}\big) \cap \mathbf{Mat}_{g_2 \times g_1}(\mathbb{Q}),$$

and the latter equals $\{\mathbf{M} \in \rho_a(Hom(A_1, A_2)); \mathbf{M} = \mathbf{M}^\sigma\}$. This follows from [3, Theorem 66].

3.2 Polarizations and Riemann's relations

For complex tori $T = \mathbb{C}^g/\Lambda$ one has a canonical identification $H^2(T, \mathbb{Z}) = \bigwedge^2(\Lambda, \mathbb{Z})$ (integer valued alternating forms). Hence, the image $c_1(\mathcal{L}) \in H^2(T, \mathbb{Z})$ of a line bundle $\mathcal{L} \subset Pic(T)$ with respect to the first Chern map

$$c_1 : Pic(T) \longrightarrow H^2(T, \mathbb{Z})$$

can be viewed as an alternating form E on Λ. Of course, the form E extends uniquely to $\Lambda \otimes \mathbb{R} = \mathcal{T}_A(\mathbb{C})$. By definition, the image of the first Chern map is the *Neron Severi group* $NS(T)$. For its rank it holds that $\rho(T) \le h^{1,1}(T) = g^2$. Conversely, it can be shown that $E \in H^2(T, \mathbb{Z})$ is in the image $c_1(Pic(T))$ of the first Chern map if $E(-, -) = E(i \cdot -, i \cdot -)$ (cf. [1], Proposition 2.1.6.). In this situation, E gives rise to a unique Hermitian form

$$H(v, w) = E(iv, w) + i \cdot E(v, w)\big(v, w \in \mathcal{T}_A(\mathbb{C})\big).$$

Thereby, $E = c_1(\mathcal{L})$ results from an ample line bundle $\mathcal{L} \in Pic(T)$ iff H is positive-definite (cf. [1, Sect. 4.1]). Summarizing, a torus T is algebraic iff there exists E, H as above with H positive-definite. Then $T = A(\mathbb{C})$ for a complex Abelian variety A. A pair (T, H) resp. (A, H) is called a *polarization of* T resp. a *polarization of* A, and one speaks of a *polarized Abelian variety*. In this situation, $NS(A)$ is identified with $NS(T)$.

There is another way to decide whether $E \in \bigwedge^2(\Lambda, \mathbb{Z})$ arises from a line bundle $\mathcal{L}$. Namely, letting $e_1, \dots, e_g$ be a basis of $\mathbb{C}^g$ and $\omega_1, \dots, \omega_{2g}$ be a $\mathbb{Z}$-basis of Λ, one can form the *period matrix* $\Pi = (p_{ij}) \in \mathbf{Mat}_{g\times 2g}(\mathbb{C})$, $1 \le i \le g, 1 \le j \le 2g$, which is defined by $\omega_j = \sum p_{ij} e_i$. Its conjugate Π^σ is nothing but (p_{ij}^σ). Moreover, we let $\mathbf{M} \in \mathbf{Mat}_{2g\times 2g}(\mathbb{Z})$ be the matrix representing E.

Proposition 3.1 *With preceding notations, the following holds:*

(1) $E \in NS(T) \Longleftrightarrow i\Pi\mathbf{M}^{-1}(\Pi^\sigma)^t$ *defines a Hermitian form on* $\mathbb{C}^g$, *whereas* $\Pi\mathbf{M}^{-1}\Pi^t = 0$.
(2) $E \in NS(T)$ *is an ample class* $\Longleftrightarrow$ (1) *holds and* $i\Pi M^{-1}(\Pi^\sigma)^t$ *defines a positive-definite Hermitian form on* $\mathbb{C}^g$.
(3) *If* (1) *is satisfied then* $2i \cdot (\Pi^\sigma\mathbf{M}^{-1}\Pi^t)^{-1}$ *is the matrix of* H *with respect to* $e_1, \dots, e_g$.

The identities in (1) are called *Riemann's relations*. These relations result from the following more general fact.

Lemma 3.2 *Let* Λ *be a lattice in* $\mathbb{C}^g$. *Let* $e_1, \dots, e_g$ *be a basis of* $\mathbb{C}^g$ *and* $\omega_1, \dots, \omega_{2g}$ *be a* $\mathbb{Z}$-*basis of* Λ. *Consider an alternating form* $E \in \bigwedge^2(\Lambda, \mathbb{Z})$ *and define*

$$H : \mathbb{C}^g \times \mathbb{C}^g \longrightarrow \mathbb{C}$$

by $\mathbb{R}$-*linear extension of* $H(v, w) = E(iv, w) + iE(v, w)$, $v, w \in \Lambda$. *Denote by* $\mathbf{M}$ *the skew-symmetric matrix representing* E *with respect to the chosen basis, and suppose that* $\mathbf{M}$ *is non-singular. Then* H *is Hermitian iff* $\Pi^\sigma\mathbf{M}\Pi^t = 0$. *In this case,* H *is given by the matrix* $2i(\Pi^\sigma\mathbf{M}^{-1}\Pi^t)^{-1}$.

For the proof of Proposition 3.1 and Lemma 3.2 we refer to [1, Sect. 4.2.].

Lemma 3.3 *Let* (A, H) *be a polarized complex Abelian variety such that* A *is the complexification of an Abelian variety* A' *over* $\mathbb{R}$. *Consider a proper, but nontrivial closed Lie subgroup* $G \subset A(\mathbb{R})^o$. *Define* $\mathcal{T}_G \subset \mathcal{T}_A(\mathbb{R})$ *to be the tangent space of* G *at the unit element. Then there exists a basis* $\omega_1, \dots, \omega_g$ *of* $\Lambda \cap \mathcal{T}_A(\mathbb{R})$ *such that* $\Lambda_G := \Lambda \cap \mathcal{T}_G = \sum_{1\le k\le \dim G} \mathbb{Z}\omega_k$. *Writing* $e_1 = \omega_1, \dots, e_g = \omega_g$ *and letting* $\mathbf{Z} \in \mathbf{Mat}_{g\times g}(\mathbb{R})$ *be such that the elements* $\omega_{g+k} = i\mathbf{Z}e_k$ *form a basis of* $\Lambda \cap i \cdot \mathcal{T}_A(\mathbb{R})$, *the following holds: Let* $\mathbf{M}$ *denote the matrix of the* $\mathbb{R}$-*linear mapping* $E = \operatorname{Im} H$ *with respect to the basis* $\omega_1, \dots, \omega_{2g}$. *Then* H *is represented by* $2i(\Pi^\sigma\mathbf{M}^{-1}\Pi^t)^{-1}$, *where* Π *is the matrix* $(\mathbf{1}_g, \mathbf{Z})$.

Proof By the structure theorem for finitely generated Abelian groups, there is a basis $\omega_1, \dots, \omega_g$, $g = \dim A$, of $\Lambda \cap \mathcal{T}_A(\mathbb{R})$ such that $\Lambda_G = \sum_{1\le k\le \dim G} \mathbb{Z}n_k\omega_k$ for some $n_k > 0$. As $\Lambda_G \otimes \mathbb{R} = \mathcal{T}_G$, we have indeed $n_k = 1$ or $= 0$. That is, $n_k = 1$ for $1 \le k \le \dim G$ and $n_k = 0$ otherwise. The rest is clear by Proposition 3.1 and Lemma 3.2. □

Let H be a polarization of A, where A is as in Lemma 3.3. Then the Hermitian form

$$H^\sigma(v, w) := H(v^\sigma, w^\sigma)^\sigma (v, w \in \mathcal{T}_A(\mathbb{C})) \tag{3.5}$$

is a polarization, too, since $E^\sigma = \operatorname{Im} H^\sigma$ sends $\Lambda \times \Lambda$ to $\mathbb{Z}$. Hence, after eventually substituting H by $H + H^\sigma$, we thus achieve that H is compatible with σ, i.e. $H(v^\sigma, w^\sigma) = H(v, w)^\sigma$.

Lemma 3.4 *If (3.5) holds then H is represented by a symmetric matrix $\mathbf{N} \in \mathbf{Mat}_{g\times g}(\mathbb{R})$ with the property that*

$$\mathbf{NZ} \subset \mathbf{Mat}_{g\times g}(\mathbb{Z}). \tag{3.6}$$

Proof That $\mathbf{N}$ is real is due to the special form of our matrix $\Pi = (\mathbf{1}_g, \mathbf{Z})$. And for (3.6) one has to combine '$\operatorname{Im} H(\Lambda \times \Lambda) \subset \mathbb{Z}$' with the fact that $\omega_{k_1} = e_{k_1} \in \mathcal{T}_A(\mathbb{R})$ for $k_1 \in \{1, \dots, g\}$, whereas $\omega_{k_2} \in i \cdot \mathcal{T}_A(\mathbb{R})$ if $k_2 \in \{g+1, \dots, 2g\}$. □

The last lemma has the following conceptual interpretation. In the formulation, $NS(A')$ is $c_1(Pic(A'))$, where $Pic(A') \subset Pic(A)$ is the subgroup of line bundles which are defined over $\mathbb{R}$.

Lemma 3.5 *An element $c_1(H) \in NS(A)$ lies in $NS(A')$ precisely when it is represented by a real matrix as in the previous lemma.*

Proof See [3, Remark 64]. □

Lemmas 3.3 and 3.4 imply the following result.

Proposition 3.6 *Let A be a complex Abelian variety defined over $\mathbb{R}$. Then there is an isogeny*

$$\psi : A \longrightarrow A_*$$

such that A_* and ψ are also defined over the reals and satisfying the following conditions:

(1) *The kernel Λ_* of $\exp_{A_*}$ decomposes as $\Lambda_* = (\Lambda_* \cap \mathcal{T}_A(\mathbb{R})) \oplus (\Lambda_* \cap i \cdot \mathcal{T}_A(\mathbb{R}))$.*
(2) *Letting $e_1 = \omega_1, \dots, e_g = \omega_g$ be a basis of $\Lambda_* \cap \mathcal{T}_A(\mathbb{R})$ and $\mathbf{Z} \in \mathbf{Mat}_{g\times g}(\mathbb{R})$ with the property that $\omega_{g+k} = i \cdot \mathbf{Z}e_k$ is a basis of $\Lambda_* \cap i \cdot \mathcal{T}_A(\mathbb{R})$ for $k \in \{1, \dots, g\}$, there is a polarization H and a matrix $\mathbf{C} \in \mathbf{Mat}_{g\times g}(\mathbb{Z})$ such that H is represented by $2(\mathbf{CZ}^t + \mathbf{ZC}^t)^{-1}$, whereas $E = \operatorname{Im} H$ is given by $\left(\begin{smallmatrix} 0 & \mathbf{C} \\ -\mathbf{C}^t & 0 \end{smallmatrix}\right)$.*
(3) $\chi_{\mathbb{Q}}(vol(A)) = \chi_{\mathbb{Q}}(vol(A_*))$.

Proof With notations as in Sect. 3.1 (8), $2\Lambda \subset \Lambda_*$. Thus, we receive an isogeny of tori

$$\psi : T \longrightarrow T_* = \mathcal{T}_A(\mathbb{C})/\Lambda_*$$

which corresponds to an isogeny $A \longrightarrow A_*$ of Abelian varieties. Moreover, σ commutes with ψ and therefore ψ is defined over $\mathbb{R}$ (cf. Sect. 3.1 (9)). This is (1). As to (2), the existence of H is provided by Lemma 3.4. Using the special form of the period matrix Π associated to Λ_*, one derives that $E = \operatorname{Im} H$ is in fact represented by a matrix $\left(\begin{smallmatrix} 0 & \mathbf{C} \\ -\mathbf{C}^t & 0 \end{smallmatrix}\right)$. In this situation, Lemma 3.2 yields

$$\mathbf{N} = \begin{pmatrix} \mathbf{1}_2 & -i\mathbf{Z} \end{pmatrix} \begin{pmatrix} 0 & \mathbf{C} \\ -\mathbf{C}^t & 0 \end{pmatrix} \begin{pmatrix} \mathbf{1}_2 \\ i\mathbf{Z}^t \end{pmatrix},$$

and the claim follows from a direct calculation. Finally, (3) results from Sect. 3.1 (9). □

Remark 3.7 Let T be a torus with period matrix $\Pi = (\mathbf{1}_2, \mathbf{Z}')$ where $\mathbf{Z}' = \left(\begin{smallmatrix}\alpha_1 & \alpha_2\\ \beta_1 & \beta_2\end{smallmatrix}\right)$ is not assumed to have purely imaginary entries. Let

$$\begin{pmatrix} 0 & a & c_{11} & c_{12}\\ -a & 0 & c_{21} & c_{22}\\ -c_{11} & -c_{21} & 0 & c\\ -c_{12} & -c_{22} & -c & 0 \end{pmatrix}$$

be an integer valued matrix representing a polarization $E = \operatorname{Im} H$ with respect to $\omega_1, \ldots, \omega_4$. Then Riemann's relations are equivalent to

$$a + c_{21}\alpha_1 + c_{11}\alpha_2 + c_{22}\beta_1 - c_{12}\beta_2 + c \cdot \det(\mathbf{Z}') = 0.$$

The reader is referred to [1, Sect. 2.6 (4)].

We will also use the following lemma.

Lemma 3.8 *Let A' be a real Abelian surface such that $T = A(\mathbb{C})$ has period matrix $\Pi = (\mathbf{1}_2, i \cdot \mathbf{Z})$. Then $\rho(A) = \rho(A') + \chi_{\mathbb{Q}}(vol(A))$.*

Proof We know that an element $H \in NS(A)$ lies in $NS(A')$ iff it is represented by a 2×2-matrix $\mathbf{N}$ with respect to Π such that $\operatorname{Im} \mathbf{N} = 0$. Now, in general $\operatorname{Im} \mathbf{N}$ is skew-symmetric: $\operatorname{Im} \mathbf{N} = \left(\begin{smallmatrix}0 & a\\ -a & 0\end{smallmatrix}\right)$. As $\operatorname{Im} H(e_1, e_2) = e_1(\operatorname{Im} \mathbf{N})e_2 \in \mathbb{Z}$, it follows that $a \in \mathbb{Z}$. On the other hand, writing $i \cdot \mathbf{Z} = (\omega_3, \omega_4)$, it holds that

$$-a \cdot \det(\mathbf{Z}) = -\omega_3^t(\operatorname{Im} \mathbf{N})\omega_4 = \omega_3^t(\operatorname{Im} \mathbf{N})\omega_4^\sigma = \operatorname{Im} H(\omega_3, \omega_4) \in \mathbb{Z}.$$

Hence, $\operatorname{Im} \mathbf{N} \neq 0$ if only if $\det(\mathbf{Z}) \in \mathbb{Q}$. Conversely, if $\det(\mathbf{Z}) = p/q \in \mathbb{Q}$ then $\left(\begin{smallmatrix}0 & q\\ -q & 0\end{smallmatrix}\right)$ gives rise to an element in $NS(A) \setminus NS(A')$. The result follows. □

3.3 The algebra of endomorphisms

We begin by recalling some basic facts concerning Rosati involutions. To this end, let A be a complex Abelian variety and $H \in Pic(A)$ a polarization. H then induces an isogeny $\phi_H : A \longrightarrow \hat{A}$ to the dual Abelian variety $\hat{A}$, which indeed only depends on the class of H in $NS(A)$. Moreover, for $\phi \in F = End(A) \otimes \mathbb{Q}$ we receive an element $\phi' = \phi_H^{-1} \circ \hat{\phi} \circ \phi_H \in F$, and one checks that $\phi \mapsto \phi'$ defines an anti-involution on F, the *Rosati involution* associated to H.

Now, consider a $L \in NS(A)$. Upon letting $\Phi_H(L) = \phi_L^{-1} \circ \phi_H$, one receives a monomorphism

$$\Phi_H : NS(A) \otimes \mathbb{Q} \longrightarrow F$$

of $\mathbb{Q}$-vector spaces with image $End^s(A) \otimes \mathbb{Q}$, where $End^s(A)$ is the subalgebra of elements fixed by the Rosati involution (cf. [1, Sect. 5.2]). Next, we let $T = A(\mathbb{C})$ and write $T = \mathbb{C}^g/\Lambda$ with a lattice $\Lambda \subset \mathbb{C}^g$. Let

$$\rho_a : F \longrightarrow \mathbf{Mat}_{g\times g}(\mathbb{C})$$

be the analytic representation with respect to $\Lambda \subset \mathbb{C}^g$. For a $\phi \in F$, $\chi_\phi(t) = \det(t - \rho_a(\phi))$ is said to be the *characteristic polynomial of* ϕ. Writing $\chi_\phi(t) = \sum_{0=k}^{g}(-1)^k a_k t^{g-k}$, we define the *rational trace* $\mathrm{tr}(\phi)$ to be $2\,\mathrm{Re}\, a_1$. The association $(\phi, \psi) \mapsto \mathrm{tr}(\phi \circ \psi')$ is then a positive-definite bilinear form on F, and $'$ is said to be a *positive involution* (cf. [1, Sect. 5.1]).

In the special situation where A is simple, F will be a skew-field. Write K for its center. Letting $d = \sqrt{[F : K]}$, it follows that each $\phi \in F$ satisfies a polynomial $\sum_{0=k}^{d}(-1)^k b_k t^{g-k}$, and the trace can be defined in an abstract fashion as $\mathrm{tr}(\phi) = b_1$. We remark that both definitions coincide (cf. [1, Sect. 5.5]).

From now on let A be the complexification of a real surface A'. Then $\hat{A}$ is the complexification of $\widehat{A'}$, and for an element $L \in NS(A')$ we have that $\phi_L \in F$ is defined over $\mathbb{R}$. This applies in particular to polarizations H of A as in Lemma 3.4. Thus, if $L, H \in NS(A')$ then $\Phi_H(L)$ is defined over $\mathbb{R}$. Altogether, we receive a commutative diagram

$$
\begin{tikzcd}
NS(A) \otimes \mathbb{Q} \arrow[r, two heads, "\Phi_H"] & End^s(A) \otimes \mathbb{Q} \arrow[r] & F = End(A) \otimes \mathbb{Q} \\
NS(A') \otimes \mathbb{Q} \arrow[u] \arrow[rr] & & End(A') \otimes \mathbb{Q} \arrow[u]
\end{tikzcd}
\tag{3.7}
$$

where all arrows are injective.

3.4 Elliptic curves

We collect some results on elliptic curves which will be needed in context of Alternative 1a and 3. We consider the *Weil restriction* $\mathcal{N}_{\mathbb{C}/\mathbb{R}} : \mathbf{Var}_{\mathbb{C}} \longrightarrow \mathbf{Var}_{\mathbb{R}}$. By definition, the functor $\mathcal{N}_{\mathbb{C}/\mathbb{R}}$ is the left adjoint of '$- \times_{\mathrm{spec}\,\mathbb{R}} \mathrm{spec}\,\mathbb{C}$'. In other words, for all $V' \in \mathbf{Var}_{\mathbb{R}}$ and all $A \in \mathbf{Var}_{\mathbb{C}}$ there is a natural identification of sets of morphisms

$$Mor_{\mathbf{Var}_{\mathbb{R}}}\big(V', \mathcal{N}_{\mathbb{C}/\mathbb{R}}(A)\big) = Mor_{\mathbf{Var}_{\mathbb{C}}}(V, A)$$

where $V = V' \times_{\mathrm{spec}\,\mathbb{R}} \mathrm{spec}\,\mathbb{C}$. One can prove that $\mathcal{N}_{\mathbb{C}/\mathbb{R}}(A) \times_{\mathrm{spec}\,\mathbb{R}} \mathrm{spec}\,\mathbb{C}$ is isomorphic over $\mathbb{C}$ to $A \times A^\sigma$ and that $\mathcal{N}_{\mathbb{C}/\mathbb{R}}(A)$ is a simple Abelian variety as soon as A and A^σ are simple Abelian varieties not isogenous to each other (cf. [3, Example 25]). With this we have:

Lemma 3.9 *Let B be a complex elliptic curve that is defined over $\mathbb{R}$, i.e. $B = B' \times_{\mathrm{spec}\,\mathbb{R}} \mathrm{spec}\,\mathbb{C}$ for some $B' \in \mathbf{Var}_{\mathbb{R}}$. Then B is isogenous to B^σ. Conversely, assume that B^σ and B are isogenous and with the property that $A = B \times B^\sigma$ is the complexification of $A' = \mathcal{N}_{\mathbb{C}/\mathbb{R}}(B) \in \mathbf{Var}_{\mathbb{R}}$. If B admits CM then A' is not simple; if A' is simple then B and its conjugate are not defined over $\mathbb{R}$.*

Proof Write $B = \mathbb{C}/\Lambda$ with $\Lambda = \Lambda^\sigma$. According to [3], $B^\sigma = \mathbb{C}/\Lambda^\sigma$. This gives the first claim. For the second, assume that $B = \mathbb{C}/\Lambda$ and $B^\sigma = \mathbb{C}/\Lambda^\sigma$ are isogenous. After a change of complex coordinates we may suppose that $\Lambda = \mathbb{Z} + \mathbb{Z}\tau$ for a complex number τ contained in an imaginary quadratic number field. In this situation, $\Lambda_* = \mathbb{Z} + d \cdot i \cdot \mathrm{Im}\,\tau\mathbb{Z}$ is contained in Λ for a suitable integer $d > 0$. This yields an isogeny $B_* = \mathbb{C}/\Lambda_* \longrightarrow B$, where B_* is defined over the reals. By the universal property of the Weil restriction, we receive a nontrivial morphism $B'_* \longrightarrow A'$ which is not surjective. Finally, suppose that, say B is the complexification of a $B' \in \mathbf{Var}_{\mathbb{C}}$. Then the diagonal embedding Δ induces a morphism

$B \xrightarrow{\Delta} B \times B \xrightarrow{\text{isog.}} \mathcal{N}_{\mathbb{C}/\mathbb{R}}(B)$, and a nontrivial arrow $B' \longrightarrow A'$ over $\mathbb{R}$ is obtained. But this is impossible, because A' is simple. So, neither B nor B^σ are defined over $\mathbb{R}$. Hence the claim. □

Lemma 3.10 *B admits CM if and only if B^σ does.*

Proof Clear when writing $B = \mathbb{C}/\Lambda$, $\Lambda = \mathbb{Z} + \mathbb{Z}\tau$ and $B^\sigma = \mathbb{C}/\Lambda^\sigma$ with $\tau \in K$ for an imaginary quadratic number field K. □

For sake of completeness, we also stress the following lemma. The proof follows immediately by considering the intersection form on $Pic(A)$.

Lemma 3.11 *Let A be isogenous to a product $B \times B_*$ of elliptic curves. Then $\rho(A) \geq 2$.*

4 Proof of the theorem

4.1 Preparations

We will assume throughout that A fulfills (2) in Proposition 3.6. This additional assumption is satisfied, because, as shown above, there is always an isogeny over $\mathbb{R}$ to an Abelian variety A_* such that A_* satisfies (2) and (3) in Prop. 3.6 and because the alternatives in the theorem are stable with respect to isogenies which are defined over $\mathbb{R}$. This is clear, except for Alternative 1c which will be treated separately (Sect. 4.1, second case).

4.2 Alternative 1

Let $B' \subset A'$ be an elliptic curve defined over the reals. We are going to show that in this case A is subject to Alternative 1. Let $G = B(\mathbb{R})$ where $B = B' \times_{\operatorname{spec}\mathbb{R}} \operatorname{spec}\mathbb{C}$. Let $\mathcal{T}_G \subset \mathcal{T}_A(\mathbb{R})$ be the tangent space of G at the unit element, and define Λ_G to be $\Lambda \cap \mathcal{T}_G$. By Lemma 3.3, there is a basis ω_1, ω_2 of $\Lambda \cap \mathcal{T}_A(\mathbb{R})$ such that $\Lambda_G = \mathbb{Z}\omega_1$. We write $\mathbf{Z} = \left(\begin{smallmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{smallmatrix}\right)$ (with notations as in Lemma 3.3). Note that G is a nontrivial infinite Lie group. Namely, otherwise $\mathcal{T}_B(\mathbb{C}) \subset i \cdot \mathcal{T}_A(\mathbb{R})$ by (3.2), which is no vector space closed under complex conjugation. A' is isogenous to $B' \times B'_*$ where $B'_* = A'/B'$. So, upon eventually considering $B' \times B'_*$ instead of A', we find the following lemma.

Lemma 4.1 *If A' is a product $B' \times_{\operatorname{spec}\mathbb{R}} B'_*$ of elliptic curves over $\mathbb{R}$ then, after an eventual transition to an isogenious variety, $\mathbf{Z}$ can be chosen such that $\alpha_2 = \beta_1 = 0$. In particular, A admits a principal polarization defined over the reals.*

From now on we will assume that A and Π are as in Lemma 4.1.

Case: $\frac{\alpha_1}{\beta_2} \notin \mathbb{Q}$ Let $G' \in \mathcal{G}$ with tangent space $\mathcal{T}_{G'}$, where $\mathcal{G}$ is as in the hypotheses of the theorem (in particular, G' is compact). Because of the special form of Π, it is easy to see that $\mathcal{T}_{G'} \cap \Lambda$ has a basis of the form $a_1e_1 + a_2e_2$ with integers a_1, a_2. Let $a = \frac{a_1}{a_2}$. Since $\frac{\alpha_1}{\beta_2} \in \mathbb{Q}$, there are natural numbers m_1, m_2 with the property that $a = \frac{m_1\alpha_1}{m_2\beta_2}$. It turns out that the group $\Lambda \cap (\mathcal{T}_{G'} \otimes_{\mathbb{R}} \mathbb{C})$ has rank 2; that is, $\exp_A(\mathcal{T}_{G'} \otimes_{\mathbb{R}} \mathbb{C})$ is compact, hence an elliptic curve $B(\mathbb{C})$. Thus, $\Phi(B) = G'$. We derive that Φ is bijective. Next, from Remark 3.7 and Lemma 3.8 it results that

$$\rho(A) = 4 - r(\mathbf{Z}) + \chi_{\mathbb{Q}}(vol(A)) = 4 - r(\mathbf{Z}) + \chi_{\mathbb{Q}}(\det(\mathbf{Z})). \tag{4.1}$$

Thereby, $r(\mathbf{Z}) = \mathrm{rank}(\mathbb{Z}\alpha_1 + \mathbb{Z}\beta_2)$, which by assumption equals 1. Hence,

$$\rho(A) = 4 \quad \Longleftrightarrow \quad vol(A) \in \mathbb{Q} \quad \Longleftrightarrow \quad \alpha_1^2, \beta_2^2 \in \mathbb{Q}.$$

On the other hand, [1, Sect. 5.6 (10)] asserts that $\rho(A) = 4$ precisely when A is isogenous to a power $B \times B$ of an elliptic curve with CM, namely the elliptic curve subject to the lattice $\mathbb{Z} + i\alpha_1\mathbb{Z}$. As the latter is closed under complex conjugation, it follows then from Sect. 3.1 (9), that $B = B' \times_{\mathrm{spec}\,\mathbb{R}} \mathrm{spec}\,\mathbb{C}$ with a $B' \subset A'$ (inclusion defined over $\mathbb{R}$). Finally, it is known that A is a product of elliptic curves over $\mathbb{C}$ if it is isogenous to a product of elliptic curves with complex multiplication (cf. [1, Sect. 10.6.3.]). Altogether we receive that Alternative 1a is valid iff A is not simple over $\mathbb{R}$ and $\frac{\alpha_1}{\beta_2} \in \mathbb{Q}$. In this situation, $\rho(A) = 4$ if and only if $vol(A) \in \mathbb{Q}$.

If $\rho(A) = 3$ then $vol(A) \notin \mathbb{Q}$ by (4.1) and, as just mentioned, A cannot be isogenous to a power $B \times B$ where B has CM. On the other hand, $\frac{\alpha_1}{\beta_2} \in \mathbb{Q}$ implies that the curves $\mathbb{C}/(\mathbb{Z} + i\alpha_1\mathbb{Z})$ and $\mathbb{C}/(\mathbb{Z} + i\beta_2\mathbb{Z})$ are isogenous. Thus, A is isogenous to $B \times B$ where $B = \mathbb{C}/(\mathbb{Z} + i\alpha_1\mathbb{Z})$. Also, it results that A admits RM and CM because $F = \mathbf{Mat}_{2\times 2}(End_{\mathbb{Q}}(B))$. In other words, we are in Alternative 1b.

Case: $\boldsymbol{\frac{\alpha_1}{\beta_2} \notin \mathbb{Q}}$ We will verify in three steps that this is equivalent to Alternative 1c. Firstly, it is clear that $\mathcal{B}$ is finite, whereas $\mathcal{G}$ is not, and that the two curves $B(\mathbb{C}) = \mathbb{C}/(\mathbb{Z} + i\alpha_1\mathbb{Z})$ and $B_*(\mathbb{C}) = \mathbb{C}/(\mathbb{Z} + i\beta_2\mathbb{Z})$ are not isogenous. Secondly, $\frac{\alpha_1}{\beta_2} \notin \mathbb{Q}$ implies that $r(\mathbf{Z}) = 2$, and it results from (4.1) that $\rho(A) = 2$ iff $vol(A) \notin \mathbb{Q}$, and $\rho(A) = 3$ iff $vol(A) \in \mathbb{Q}$. So, assume that $vol(A) \in \mathbb{Q}$. Then $vol(A)/\alpha_1 = \beta_2$, so that B and B_* would be isogenous. Hence, $\rho(A) = 2$ and $vol(A) \notin \mathbb{Q}$. Next,

$$2 \leq [F : \mathbb{Q}] = [End_{\mathbb{Q}}(B) : \mathbb{Q}] + [End_{\mathbb{Q}}(B_*) : \mathbb{Q}] \leq 4,$$

and all values between 2 and 4 are possible and depending on if B or B_* admits CM. Thirdly, we remember that the assumption that $\mathbf{Z}$ is a diagonal matrix $\mathrm{diag}(\alpha_1, \beta_2)$ is already a consequence of a reduction step (transition from A to an isogenous Abelian surface). Thus, let us withdraw the mentioned reduction; that is, let us start again with a pair $A_*, \mathbf{Z}_*$ that is isogenous over $\mathbb{R}$ to $A, \mathbf{Z} = \mathrm{diag}(\alpha_1, \beta_2)$, but such that we do not presuppose from the beginning that $\mathbf{Z}_*$ has diagonal form. Let us prove that A_* is still a product of non-isogenous curves over $\mathbb{R}$. To this end, we will show that $\mathbf{Z}_*$ is in fact a diagonal matrix. We proceed by contradiction. An $\mathbb{R}$-isogeny between A and A_* is necessarily represented by a matrix in $\mathbf{Gl}_2(\mathbb{Q})$. Hence, the entries of the first row of $\mathbf{Z}_*$ are rational multiples of α_1, whereas the entries of the second row are rational multiples of β_2. Precisely the same holds true for the transpose of $\mathbf{Z}_*$. Namely, by [1, Sect. 4.12 (8)] the dual $\hat{A}$ admits a period matrix $(\mathbf{1}_2, i \cdot \mathbf{Z}^t)$, and by the previous lemma A is isomorphic over $\mathbb{R}$ to $\hat{A}$. Thus, it is easy to see: If $\mathbf{Z}_*$ is no diagonal matrix then $r(\mathbf{Z}_*) = 1$, a contradiction to the above. Consequently, $\mathbf{Z}_*$ is a diagonal matrix and A_* is a product of elliptic curves.

Now, if we more generally start with an arbitrary A defined over $\mathbb{R}$, but such that A does not necessarily allow a period matrix $(\mathbf{1}_2, i \cdot \mathbf{Z})$, then it follows from (3.3): There is an isogeny $\phi : A \longrightarrow A_*$ defined over $\mathbb{R}$ from an Abelian surface A_* with period matrix $(\mathbf{1}_2, i \cdot \mathbf{Z}_*)$ such that $\deg \phi = 2^k$, where $k \leq 3$ is the degree of connectedness (cf. Sect. 3.1 (8)). Everything is proved.

4.3 Alternative 2

We again suppose that A satisfies (2) in Proposition 3.6. In the next lemma, $r(\mathbf{Z})$ is defined to be the rank of $\mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2 + \mathbb{Z}\beta_1 + \mathbb{Z}\beta_2$.

Lemma 4.2 *If A' is simple then the pairs α_1, α_2 and β_1, β_2 satisfy no nontrivial relation over $\mathbb{Z}$. Moreover, $r(\mathbf{Z}) \leq 3$.*

Proof By way of contradiction, assume α_1 and α_2 to be linear dependent over $\mathbb{Z}$. It follows that $i\mathbb{R}\omega_2/(\Lambda \cap i\mathbb{R}\omega_2)$ is compact. Hence, $B(\mathbb{C}) = \exp_A(\mathbb{C}\omega_1)$ is compact, and therefore an elliptic curve contained in $T = A(\mathbb{C})$. Section 3.1 (7) implies that $B(\mathbb{C})$ in fact results from a $B' \in \mathbf{Var}_{\mathbb{R}}$, a contradiction. Likewise, β_1 and β_2 are $\mathbb{Z}$-linear independent. So far we have proved that $r(\mathbf{Z}) \geq 2$. The estimate $r(\mathbf{Z}) \leq 3$ follows from Remark 3.7. □

Next, we study the endomorphism algebra of A. We will need the general classification of skew-fields F with positive (anti-)involutions $'$ arising from simple polarized Abelian varieties A. To this end, start with an arbitrary skew-field F which has dimension ≤ 4 over $\mathbb{Q}$ and with positive (anti-)involution $'$. Let K be the center of F, and $\rho = [F^s : \mathbb{Q}]$ where F^s is the subalgebra of elements fixed by the involution. Then there are the following possibilities according to *Albert's classification* (cf. [1], Proposition 5.5.7, e.g.):

(1) F is a totally real number field: $[F : K] = 1$, $[K : \mathbb{Q}] = \rho \leq 2$, and $F = F^s$.
(2) F is a totally indefinite quaternion algebra with CM and RM: $[F : K] = 4$, $[K : \mathbb{Q}] = 1$, $\rho = 3$.
(3) F is a totally definite quaternion algebra: $[F : K] = 4$, $[K : \mathbb{Q}] = 1$, $\rho = 1$.
(4) F is of the second kind, that is, a totally complex extension of a real quadratic number field F^s: $[F : K] = 1$, $\rho = 2$.

We will apply Albert's classification to our endomorphism algebra $F = End(A) \otimes \mathbb{Q}$ and a fixed Rosati involution which results from a polarization over $\mathbb{R}$. We will examine F using the analytic representation

$$\rho_a : F \longrightarrow \mathbf{Gl}(\mathcal{T}_A(\mathbb{C})) = \mathbf{Mat}_{2\times 2}(\mathbb{C}),$$

where the right identification is with respect to the basis $e_1 = \omega_1, e_2 = \omega_2$.

Lemma 4.3 *If A is simple and if $[F : \mathbb{Q}] = 4$ then F is not defined over $\mathbb{R}$ and is an indefinite quaternion algebra.*

Proof If F were defined over the reals then $\rho_a(F) = \mathbf{Mat}_{2\times 2}(\mathbb{Q})$ according to Sect. 3.1 (9). Of course, this is absurd. Next, noting that $\rho_a(F)^\sigma = \rho_a(F)$, we infer that there exists a nontrivial element in $\mathbf{N} \in \rho_a(End(A)) \cap i \cdot \mathbf{Mat}_{2\times 2}(\mathbb{R})$. $\mathbf{N}$ is a linear isomorphism of $\mathcal{T}_A(\mathbb{C})$, since $\mathbf{N}$ necessarily results from an isogeny. Thus, we have an isogeny which interchanges the real part $A(\mathbb{R})$ with the part coming from $i \cdot \mathcal{T}_A(\mathbb{R})$, that is

$$\mathbf{N}(\mathcal{T}_A(\mathbb{R})) \subset i \cdot \mathcal{T}_A(\mathbb{R}), \mathbf{N}(i \cdot \mathcal{T}_A(\mathbb{R})) \subset \mathcal{T}_A(\mathbb{R}). \tag{4.2}$$

Moreover, $\mathbb{Q}(\mathbf{N})$ is a number field of degree ≤ 4. If $\mathbb{Q}(\mathbf{N})$ has degree 4 then (4.2) implies that $\mathbf{N}^4 = q_1\mathbf{N}^2 + q_2$ with $q_1, q_2 \in \mathbb{Q}$, and therefore $\mathbb{Q}(\mathbf{N}^2)$ has degree 2. Furthermore, it follows from Sect. 3.1 that $\mathbb{Q}(\mathbf{N}^2) = End(A') \otimes \mathbb{Q}$. By the lemma below it results then that

$r(\mathbf{Z}) = 2$, such that $\rho(A) \geq 2$ according to (4.1). Thus, if $\mathbb{Q}(\mathbf{N})$ has degree 4 then F must be of the second kind and $\rho(A) = 2$. We will assume this and will derive a contradiction in two steps:

(a) First, let us show that σ equals the Rosati involution $'$. Namely, noting that $|Gal(F/\mathbb{Q})| = 4$, it follows that otherwise $'$ and σ would generate the whole Abelian Galois group. Hence, $\mathbf{N} \cdot \mathbf{N}' \in \mathbb{Q} \cdot \mathbf{1}_2$. In fact, one easily verifies that $\mathbf{N} \cdot \mathbf{N}'$ belongs to the fixed field of σ and $'$. Consequently, $\det(\mathbf{N} \cdot \mathbf{N}')$ is a rational square, and in particular positive. On the other hand, according to [1, Sect. 2.6, (13), a)], for all $\phi \in F$ the transpose of $\rho_a(\widehat{\phi})$ equals $\rho_a(\phi)^\sigma$. With this and the definition of $'$ we find that $\det(\mathbf{N}^\sigma) = \det(\mathbf{N}')$. And since $\mathbf{N} \in i \cdot \mathbf{Mat}_{2\times 2}(\mathbb{R})$, $\det(\mathbf{N}) = -\det(\mathbf{N}')$. We derive a contradiction: $\det(\mathbf{N} \cdot \mathbf{N}') < 0$. Therefore, $\sigma = '$.

(b) By [1, Lemma 5.5.4], $\rho_a(F)$ admits an embedding $j : \rho_a(F) \longrightarrow \mathbb{C}$ such that $\sigma = '$ becomes complex conjugation. As $\mathbf{N}^\sigma = -\mathbf{N}$, we infer that $j(\mathbf{N}) = \pm i \cdot r$ for a $r > 0$. Likewise, if τ is a further element in the Abelian group $Gal(\rho_a(F)/\mathbb{Q})$ then $\tau(\mathbf{N})^\sigma = -\tau(\mathbf{N})$, and therefore $j(\tau(\mathbf{N})) = \pm i \cdot s$ for a $s > 0$. Assume that $s \neq r$. The two numbers $-r^2$, $-s^2$ reappear as the eigenvalues of $\mathbf{N}^2$, because they satisfy the minimal polynomial of $\mathbf{N}^2$. However, since $\mathbf{N}^2$ has degree 2 over $\mathbb{Q}$, it follows that $(\mathbf{N}^2 + d \cdot \mathbf{1}_2)^2 \in \mathbb{Q} \cdot \mathbf{1}_2$ for a certain rational number d. Let $\mathbf{M} \in \mathbf{Mat}_{2\times 2}(\mathbb{C})$ be such that $\mathbf{M} \cdot \mathbf{N}^2 \cdot \mathbf{M}^{-1}$ is in Jordan normal form. As $r^2 \neq s^2$, $\mathbf{M}(\mathbf{N}^2 + d \cdot \mathbf{1}_2)\mathbf{M}^{-1}$ is a diagonal matrix with entries $-r^2 + d = -s^2 + d$ along the diagonal. So, $r \neq s$ implies $r = s$. Hence, $r = s$ by virtue of logic. It results that $\mathbf{N}^2$ is fixed by $Gal(\rho_a(F)/\mathbb{Q})$. Consequently, it lies in $\mathbb{Q} \cdot \mathbf{1}_2$, a contradiction.

Altogether, $\mathbb{Q}(\mathbf{N})$ has degree 2, and (4.2) implies that $\mathbf{N}^2 = d \cdot \mathbf{1}_2$ for a rational number d. Consequently, $\det(\mathbf{N}) \pm d$. But, it follows from (4.2) that $\mathbf{N} = \mathbf{Z} \cdot \mathbf{D}$ for a matrix $\mathbf{D}$ with rational entries. We infer that $vol(A) = |\det(\mathbf{Z})| \in \mathbb{Q}$. Finally, we already know that $r(\mathbf{Z}) = 2$, and it results from (4.1) that $\rho(A) = 3$. So, A has quaternion multiplication. The lemma is proved. □

Lemma 4.4 *If A is simple and if there exists a $\phi \in End(A') \otimes \mathbb{Q} \setminus \mathbb{Q} \cdot id_{A'}$ such that $\phi^2 \in \mathbb{Q} \cdot id_{A'}$ then $r(\mathbf{Z}) = 2$.*

Proof The hypotheses imply that there is a matrix $\mathbf{N} \in \rho_a(End(A')) \subset \mathbf{Mat}_{2\times 2}(\mathbb{Q}) \setminus \mathbb{Q} \cdot \mathbf{1}_2$ such that $\mathbf{N}^2 \in \mathbb{Q} \cdot \mathbf{1}_2$. Hence, $\mathbf{N} = \left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right)$. Therefore, also $\mathbf{N}' = \left(\begin{smallmatrix} 2a & b \\ c & 0 \end{smallmatrix}\right) \in End(A') \otimes \mathbb{Q}$. Noting that $cb = \det \mathbf{N}' \neq 0$ as F is a skew-field, the claim follows from Lemma 4.2 and because of the special form of our period matrix: $(\mathbf{1}_2, i \cdot \mathbf{Z})$. □

Lemma 4.5 *If A is simple and if F is not defined over $\mathbb{R}$ then there exists an Abelian variety A_* which is isogenous over $\mathbb{R}$ to A and having period matrix $(\mathbf{1}_2, i \cdot \mathbf{Z}_*)$ such that*

$$\mathbf{Z}_* = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix}.$$

Here, $\alpha, \beta, \gamma \in \mathbb{R}$ and $\alpha^2 + \beta\gamma \in \mathbb{Q}$.

Proof Let $\mathbf{N} \in \rho_a(F)$ have purely imaginary entries. As above, we find $\mathbf{N}^2 \in \mathbb{Q} \cdot \mathbf{1}_2$. Since A is simple, $\mathbf{N} \notin i\mathbb{R} \cdot \mathbf{1}_2$. So,

$$\mathbf{N} = i \cdot \begin{pmatrix} a & b \\ c & -a \end{pmatrix}. \tag{4.3}$$

In particular, $\det(\mathbf{N}) = -a^2 - cb \in \mathbb{Q}$. This gives the result. □

Lemma 4.6 *If A is simple and if $[F:\mathbb{Q}]=4$ then $End(A')\otimes\mathbb{Q}$ is a quadratic number field.*

Proof Assume that $End(A')=\mathbb{Z}$. Recalling that $\rho_a(F)^\sigma=\rho_a(F)$, we infer that $\rho_a(F)\cap i\cdot\mathbf{Mat}_{2\times 2}(\mathbb{R})$ has dimension ≥ 2 over $\mathbb{Q}$. So, there are $\mathbb{Q}$-linearly independent elements $\mathbf{N},\mathbf{M}\in\rho_a(F)\cap i\cdot\mathbf{Mat}_{2\times 2}(\mathbb{R})$ as in (4.3). Since $\mathbf{N}^2$ and $\mathbf{N}\cdot\mathbf{M}$ are still $\mathbb{Q}$-linearly independent, it follows that $\mathbf{N}\cdot\mathbf{M}$ has real entries, but is not contained in $\mathbb{Q}\cdot\mathbf{1}_2$. By virtue of Sect. 3.1 (9), $End(A')\neq\mathbb{Z}$. The contradiction finishes the proof. □

Lemma 4.7 *If A is simple and if $[F:\mathbb{Q}]=4$ then $End(A')$ is an order in a totally real quadratic number field. We have then $\rho(A')=2$ and $vol(A)\in\mathbb{Q}$. Moreover, there is a further embedding $\mathcal{O}\subset End(A)$ of an order in totally real number field which is not contained in $End(A')$.*

Proof We already know that F is an indefinite quaternion algebra. By what was said before, there is a $\mathbf{M}\in\rho_a(End(A))\cap i\cdot\mathbf{Mat}_{2\times 2}(\mathbb{R})$ and a $\mathbf{N}\in\rho_a(End(A'))$ with the property that $\mathbf{M}^2,\mathbf{N}^2\in\mathbb{Q}\cdot\mathbf{1}_2$ and $\det(\mathbf{M})\in\mathbb{Q}$. Since $\mathbf{M}=\mathbf{Z}\cdot\mathbf{D}$ for a matrix $\mathbf{D}$ with rational entries, we find that $vol(A)$ is rational. So, by Lemma 3.8 and Albert's classification, $\rho(A')=\rho(A)-1=2$. This in turn implies $\rho(A')=[End(A')\otimes\mathbb{Q}:\mathbb{Q}]$, and we infer using (3.7) that $\mathrm{tr}(\mathbf{N}^2)=\mathrm{tr}(\mathbf{N}\cdot\mathbf{N}')>0$; here tr is as in Sect. 3.3. However, if $\mathbf{N}^2=d\cdot\mathbf{1}_2$ then $\chi_{\mathbf{N}^2}(t)=(t-d)^2$. Therefore, $d>0$ by definition of tr. Finally, as an abstract algebra F equals $\mathbb{Q}+\mathbb{Q}i+\mathbb{Q}j+\mathbb{Q}ij$ with $i^2<0<j^2$ and $ij=-ji$. Therefore, $(ij)^2>0$, and there is a further real order in F. Everything is proved. □

Lemma 4.8 *If A is simple and if $[F:\mathbb{Q}]=2$ then $End(A)$ is defined over $\mathbb{R}$ and is an order in a real number field. Moreover, $vol(A)\notin\mathbb{Q}$.*

Proof Suppose that $End(A)$ is not defined over $\mathbb{R}$. Then $\rho(A')=1$ by (3.7). Furthermore, we may assume that $\mathbf{Z}$ is as in Lemma 4.5. It follows from Lemma 3.5 and Remark 3.7 that the subspace of vectors $(c_{11},c_{12},c_{21},c_{22})\in\mathbb{Q}^4$ satisfying

$$c_{21}\alpha+c_{11}\gamma+c_{22}\beta-c_{12}\alpha=0$$

can be embedded into $NS(A')\otimes\mathbb{Q}$. But this subspace has dimension ≥ 2, because α, β and γ fulfill at least one nontrivial relation over $\mathbb{Z}$. The contradiction shows that $End(A')=End(A)$. Next, it results from Albert's classification that F is a real number field and that the Rosati involution is trivial. Thus, $\rho(A)=\rho(A')=2$ according to (3.7). Finally, (4.1) implies that $vol(A)\notin\mathbb{Q}$. □

The above results cover Alternatives 2a–2c.

4.4 Alternative 3

We suppose A' simple and A not simple. Then A is isogenous over $\mathbb{C}$ to a product $B\times B_*$. Thus, we have a nontrivial morphism $\pi:A\longrightarrow B$. By the universal property of the Weil restriction (Sect. 3.4), π yields an arrow $\mathcal{N}_{\mathbb{C}|\mathbb{R}}(\pi):A'\longrightarrow\mathcal{N}_{\mathbb{C}|\mathbb{R}}(B)$ in $\mathbf{Var}_{\mathbb{R}}$. As π is not trivial, so isn't $\mathcal{N}_{\mathbb{C}|\mathbb{R}}(\pi)$. Hence, it results from the simplicity of A' that $\mathcal{N}_{\mathbb{C}|\mathbb{R}}(\pi)$ is an isogeny. Applying $'-\times_{\operatorname{spec}\mathbb{R}}\operatorname{spec}\mathbb{C}'$, we receive an isogeny

$$\phi:A\longrightarrow\mathcal{N}_{\mathbb{C}/\mathbb{R}}(B)=B\times B^\sigma$$

that is defined over $\mathbb{R}$. Lemma 3.9 implies now that B and B^σ cannot have CM.

If B and B^σ are not isogenous then F is the product ring $\mathbb{Q}\times\mathbb{Q}$, and $[F:\mathbb{Q}]=2$. Also, according to Lemma 3.11, $\rho(A)\geq 2$. Therefore, $\rho(A)=2$. Next, note that F is then not defined over $\mathbb{R}$. In fact, A' is simple and thus, otherwise, F would be a simple algebra, too. Thus, $\rho(A')=1$, and it results from (4.1) that $vol(A)\in\mathbb{Q}$. This is Alternative 3a.

Finally, let B and B^σ be isogenous. Lemma 3.9 asserts that these curves are not defined over $\mathbb{R}$. Write $B(\mathbb{C})=\mathbb{C}/\Lambda$ with a lattice $\Lambda=\mathbb{Z}+\tau\mathbb{Z}$ such that τ is contained in the upper half plane. Let $\alpha\in\mathbb{C}^*$ be such that $\alpha,\alpha\tau\in\Lambda^\sigma=\mathbb{Q}+\tau^\sigma\mathbb{Q}$. It results that $|\alpha|^2=\alpha\alpha^\sigma\in End(\Lambda)$. Write $\alpha=a+b\tau^\sigma$. Let's assume first that $b=0$. Then it follows at once that $\operatorname{Re}\tau\in\mathbb{Q}$, and, as a consequence of Sect. 3.1 (9), B is defined over $\mathbb{R}$. Contradiction. So, $b\neq 0$ and $\operatorname{Re}\tau\notin\mathbb{Q}$. Since B admits no CM, $|\alpha|^2\in\mathbb{Z}$. Note that $\mathbb{Z}+\alpha\mathbb{Z}\subset\Lambda^\sigma$ is a sublattice with conjugate lattice $\mathbb{Z}+\alpha^\sigma\mathbb{Z}\subset\Lambda$. Hence, after eventually replacing Λ by $\mathbb{Z}+\alpha\mathbb{Z}$, we find that A has period matrix $\left(\mathbf{1}_2, \left(\begin{smallmatrix}\alpha & 0\\ 0 & \alpha^\sigma\end{smallmatrix}\right)\right)$ with $|\alpha|^2\in\mathbb{Q}, \operatorname{Re}\alpha\notin\mathbb{Q}$. It results from Remark 3.7 that $\rho(A)=3$. As $End(A')$ has at most rank 2 over $\mathbb{Z}$, we have $\rho(A')=2$ and $vol(A)\in\mathbb{Q}$. Furthermore, the diagram in (3.7) implies that $End(A')\subset End^s(A)$. The same way as in the proof of Lemma 4.7 one verifies now that $End(A')$ is an order in a real quadratic number field. Finally, F is clearly isomorphic to $\mathbf{Mat}_{2\times 2}(\mathbb{Q})$. Thus, we are in Alternative 3b.

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